

Intermittency in Turbulence: computing the scaling exponents in shell models

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We discuss a stochastic closure for the equation of motion satisfied by multi-scale correlation functions in the framework of shell models of turbulence. We give a systematic procedure to calculate the anomalous scaling exponents of structure functions by using the exact constraints imposed by the equation of motion. We present an explicit calculation for fifth order scaling exponent at varying the free parameter entering in the non-linear term of the model. The same method applied to the case of shell models for Kraichnan passive scalar provides a connection between the concept of zero-modes and time-dependent cascade processes.

I. INTRODUCTION

Since the fundamental work by Kolmogorov, it has been widely recognized that a consistent theory for the statistical properties of turbulence should quantitatively explain intermittency. In the last ten years many important steps have been done in providing a consistent picture of intermittency in turbulence. First, careful designed experimental measurements and a systematic way to analyze data have shown the universal feature of intermittency [1, 2]. Second, a well defined theory has been proposed to compute anomalous scaling for a class of *linear* problem, i.e. the case of Kraichnan passive scalar [3]. In the latter case, the notion of *zero-modes* provided a theoretical framework for many fundamental properties of intermittency. Yet, we are still looking for defining a suitable strategy for a quantitative computations of intermittency in the full non linear problem, namely the Navier-Stokes equations. The problem of anomalous scaling must be divided in two steps. First, we need to clean it from all unwanted difficulties, trying to focus on the main physical mechanism leading to small-scales intermittency and to its connections with the non-linear structure of the equation of motion. This is the main goal of this article. We show that the anomalous scaling of small-scales velocity fluctuations of a shell model of turbulence can be derived from the equation of motion. The result is based on a stochastic closure. A second, more ambitious goal, is to extend this results to the full complexity of Navier-Stokes equations. Some comments on the latter problem are also proposed in the conclusions.

Let us make a few general comments on the nature of the problem we are facing. We are interested in the (universal) features of the statistical properties of the velocity field, $\mathbf{v}(\mathbf{x}, t)$, in a homogeneous and isotropic turbulent flow. Experimental data and theoretical ideas suggest that these universal properties are related to velocity fluctuations at scales much smaller than outer energy input scale, L . More precisely, we want to compute the simultaneous multi-point correlation functions $C^{(n)}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = \langle v_{i_1}(\mathbf{x}_1, t) v_{i_2}(\mathbf{x}_2, t), \dots, v_{i_n}(\mathbf{x}_n, t) \rangle$ for scale separations $|\mathbf{x}_i - \mathbf{x}_j|$ much smaller than L . Our task must be performed by using the Navier-Stokes equations. This is precisely the result which has been accomplished in the case of the Kraichnan model previously referred to. The equation of motions provide a relations among the (infinite) sets of simultaneous correlation functions, $C^{(n)}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$:

$$0 = \frac{d}{dt} C^{(n)}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = \Gamma[C^{(n+1)}(\mathbf{x}_1; \mathbf{x}_2; \dots; \mathbf{x}_n)], \quad (1)$$

where we have assumed stationarity and where with Γ we denote the integro-differential linear operator derivable explicitly from the Navier-Stokes equations.

The equivalent of the above hierarchy rewritten for the case of the Kraichnan model is closed order by order in the correlation functions allowing for a perturbative calculation of the statistical properties. In the full non linear problem of the Navier-Stokes equations, one can show that (1) do not form a closed set of equations, rather it should be considered as a constrain for the complete solution. Actually, the fundamental quantities for studying intermittency in turbulence involve also temporal information from multi-time correlation functions, $C^{(n)}(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2; \dots; \mathbf{x}_n, t_n) = \langle v_{i_1}(\mathbf{x}_1, t_1) v_{i_2}(\mathbf{x}_2, t_2) \dots v_{i_n}(\mathbf{x}_n, t_n) \rangle$. Namely, we need to look for the solution of the problem:

$$\frac{\partial^k}{\partial t_1 \dots \partial t_k} C^{(n)}(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2; \dots; \mathbf{x}_n, t_n) = \Sigma[C^{(n+1)}], \quad (2)$$

with $k \leq n$ and where Σ is a functional of the time dependent correlation functions of order $n + 1$ depending on $n + 1$ velocity fields at n different times. The fundamental question we are facing is which are, if any, the physical informations we should use in order to *solve* (2). In some broad sense, being not able to solve (2) by any kind of

“brute force” attempt, we still need to understand which are the correct “order parameters” we should consider to find out a systematic way to compute a solution of (2).

We argue that a strategy to compute the solutions of the multi-time hierarchy (2) may be outlined by first finding a “physically consistent” solutions for the simultaneous hierarchy (1) which can be used as the starting point for successive approximations. For “physically consistent” we mean that the solution should respect the phenomenological constraints imposed by the Navier Stokes equations and in particular by its time-space scaling properties. This is the main idea pursued in this paper. More precisely we will discuss how far we can provide a quantitative computation of intermittency based on the following three main points: (i) we only use the constraints coming from the simultaneous equations (1); (ii) we look for the solutions of (1) by assuming that the out-of-equilibrium statistical properties of the velocity field can be obtained by a suitable time dependent stochastic process; (iii) we shall restrict ourselves to non-linear shell models [1, 4].

Having discussed in details the motivation of point (i), let us briefly comment on point (ii). Random multiplicative processes have been often used in literature as a simple mathematical tool to describe anomalous scaling properties of turbulent flows [5]. Only a few attempts have successfully linked cascade-multiplicative process with the structure of the equation of motion [6]. Recently, the concept of random multiplicative process have been enlarged by including non trivial time dynamics [7, 8]. In particular, the choice of time dynamics can be done in order to satisfy the Navier Stokes temporal scaling (in a Lagrangian reference frame). Moreover, it has been shown that time dynamics affects in non trivial way also the spatial scaling of $C^{(n)}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$ as a function of intermittency. Our strategy is to employ the statistical constraint of time dependent random multiplicative process to look for a solution of the equations (1). The theory of time dependent random multiplicative process is in its infancy and only few exact results have been obtained so far. One can wonder why we need a time dependent stochastic process as a tool to describe equal time correlation functions. The answer is that the shape of the correlation functions is strongly dependent by the time dynamics [9]. The hope is that, by using the dynamical scaling required by the Navier Stokes equations, we can already obtain a good approximations to the real solutions. Finally we want to comment on point (iii). Shell models provide the simplest model where to check our strategy and to compare our physical ideas against clean numerical simulations in the asymptotic regime of large Reynolds numbers.

Even with the approximations defined in points (i)–(iii), the problem of computing the universal anomalous scaling in turbulence is equivalent to solve a functional equation, i.e. each equation (1) defines for any order a constrain to be satisfied for the probability distribution. We will limit ourselves to the lowest, non trivial, order such as to be able to calculation analytically as much as possible.

The paper is organized in the following way.

In sect. (II) we briefly remind the basic properties of time-dependent random multiplicative process. In section (III) we address the problem of anomalous scaling in Kraichnan shell models of passive scalars. We present a re-derivation in the framework of stochastic closure of an exact result for the anomalous scaling of fourth order structure function. Thus, we are able to connect the mathematical notion of *zero-modes* with a *cascade mechanism* as given by the time-dependent multiplicative process here used. In section (IV) we extend the stochastic closure used fruitfully for the passive scalar case to the fully non-linear model. We discuss at length both similarities and differences between the two cases and we present the first attempt to calculate the fifth-order scaling exponents at varying the values of the free parameter in the shell model. Conclusions will follow in section (V).

II. TIME-DEPENDENT RANDOM MULTIPLICATIVE PROCESS

Let us first review the main ingredients of time-dependent multiplicative process (TRMP) [8]. We introduce a set of reference scales, $\ell_n = \ell_0 2^{-n}$, and a set of velocity increments at scales, ℓ_n : $\delta_n v \sim v(x + \ell_n) - v(x)$. The basic idea of random *time-independent* multiplicative process is to assume that statistical properties of $\delta_n v$ can be obtained by

$$\delta_{n+1} v = A_{n+1} \delta_n v, \quad (3)$$

where A_i are i.i.d. (independent identically distributed) random variables with a time-independent –bare– probability $P(A)$. Time independent multiplicative process as (3) have been widely used in the past to mimics the spatial distribution of velocity fluctuations in turbulence and the multi-fractal energy dissipation measure [1]. Recently, also a first attempt to match the stochastic multiplicative model with the deterministic structure of the equation of motion of a shell model of turbulence has been presented [6]. Despite the success to reproduce the cascade phenomenology, time-independent multiplicative process cannot capture the subtle complexity inter-wined in the spatial and temporal behavior of Navier-Stokes equations (in a Lagrangian reference frame). For example, multi-scale correlation functions of the kind, $\langle \delta_n v \delta_{n'} v \rangle$ are well reproduced only asymptotically for large scale separations, $n' \gg n$ [10]. The problem is that simple time-independent random multiplicative processes do not take in to account the time-dynamics, i.e.

they are not constrained by the equation of motion. To overcome this difficulties, recently, a new class of time-dependent stochastic multiplicative process (TRMP) have been proposed [7, 8]. Basically, the idea is to mimics the temporal constraints imposed by the structure of the Navier-Stokes eqs, $\partial_t v \sim v \partial v$, by requiring that the multiplicative structure (3) is satisfied for the random time interval $\tau_{n+1} = \ell_n / (\delta_n v)$. To build the temporal dynamics we proceed as follows. We extract the instantaneous multiplier, A_n , connecting the amplitudes of two velocity fluctuations at adjacent scales, $\delta_n v = A_n \delta_{n-1} v$, with a given probability $P(A)$, independent from the scale, ℓ_n , and keep it constant for a time interval $[t, t + \tau_n]$, with $\tau_n = \ell_n / (\delta_n v)$ being the local instantaneous eddy-turn-over time. Thus, for each scale ℓ_n , we introduce a time dependent random process $A_n(t)$ which is piece-wise constant for a random time intervals $[t_n^{(k)}, t_n^{(k)} + \tau_n]$, if $t_n^{(k)}$ is the time of the k th jump at scale n . The corresponding velocity field at scale n , in the time interval $t_n^{(k)} < t < t_n^{(k)} + \tau_n$, is given by the simple multiplicative rule:

$$\delta_n v(t) = A_n(t) \delta_{n-1} v(t_n^{(k)}) \quad (4)$$

What is important to notice is that at each jumping time, $t_n^{(1)}, t_n^{(2)}, \dots, t_n^{(k)}$.. only the velocity field at the corresponding shell, n is updated, i.e. information across different scales propagates with a finite speed. In this way we reproduce the phenomenology of the non-linear evolution of Navier-Stokes dynamics: $\partial_t v \propto v \partial v$. Multipliers at different scales, develop correlations through the time-dependency. From now on, we will denote with $\overline{\dots}$ averages with respect to the stochastic process and with $\langle \dots \rangle$ averages over the chaotic deterministic dynamics of the shell model.

A more detailed numerical and theoretical analysis of the statistical properties of TRMP can be found in [8]. The possibility to reproduce the *single-time* statistical properties of TRMP by a Gibbs-like measure has also been recently discussed in [9].

III. TRMP AND THE KRAICHNAN MODEL

We start our work by understanding the relationship between the Kraichnan shell model for passive scalar and time-dependent random multiplicative process. We first review the model and how the computation of the anomalous exponents can be rigorously done in this case. The model is defined as follows [11, 12]. Passive increments are described on a discrete sub-set of wave-numbers (shells), $k_n = k_0 \lambda^n$, by a complex variable $\theta_n(t)$, which satisfy the equations ($n = 1, 2, \dots, N$)

$$\left[\frac{d}{dt} + \kappa k_n^2 \right] \theta_n(t) = i[c_n \theta_{n+1}^*(t) u_n^*(t) + b_n \theta_{n-1}^*(t) u_{n-1}^*(t)] + \delta_{1n} f(t) \quad (5)$$

where the star denotes complex conjugation and $b_n = -k_n, c_n = k_{n+1}$ are chosen such as to impose energy conservation in the zero diffusivity limit. The inter-shell ratio must be taken $\lambda > 1$. Boundary conditions are defined as: $u_0 = \theta_0 = 0$. The forcing term $\delta_{1n} f(t)$ is Gaussian and delta correlated: $\langle f(t) f(t') \rangle = F_1 \delta(t - t')$. It acts only on the first shell. Kraichnan models of passive advection assume that each velocity variables, $u_n(t)$, is a complex Gaussian and white-in-time stochastic process, with a variance which scales as: $\langle u_m(t) u_n^*(t') \rangle = \delta(t - t') \delta_{nm} d_m$, $d_m = k_m^{-\xi}$. The cross-correlation between the advecting velocity variables and the passive variable can be rewritten in terms of passive correlations only, when the velocity field is a white-in-time Gaussian variable. Thus, all equations for all passive structure functions are closed [11, 12, 14]. The goal is to calculate the scaling exponents, $H(p)$ of the p -th order passive structure functions as defined by:

$$\langle |\theta_n|^p \rangle \sim k_n^{-H(p)}$$

We concentrate on the non-perturbative analytic calculation of the fourth-order structure function $P_{nn} = \langle (\theta_n \theta_n^*)^2 \rangle \propto k_n^{-\zeta_4}$ (the lowest order with non-trivial anomalous scaling). The closed equation satisfied by $P_{nq} = \langle (\theta_n \theta_n^*) (\theta_q \theta_q^*) \rangle$, is,

$$\begin{aligned} \dot{P}_{nq} = & (\delta_{1,n} E_n + \delta_{1,q} E_q) F_1 - \kappa (k_n^2 + k_q^2) P_{nq} + \\ & [-P_{nq} c_n^2 d_n ((1 + \delta_{q,n+1}) + \lambda^{\xi-2} (1 + \delta_{q,n-1})) + P_{n+1,q} c_n^2 d_n (1 + \delta_{q,n}) + P_{n-1,q} b_n^2 d_{n-1} (1 + \delta_{q,n}) + (q \leftrightarrow n)]; \end{aligned} \quad (6)$$

where $E_n = \langle \theta_n \theta_n^* \rangle = E_0 k_n^{\xi-2}$, i.e. the second order scaling exponent is given by $H(2) = 2 - \xi$. The above equation can be elegantly rewritten in the operatorial form:

$$\dot{P}_{nq} = \mathcal{I}_{nq,n'q'} P_{n'q'} + \kappa \mathcal{D}_{nq,n'q'} P_{n'q'} + \mathcal{F}_{nq}, \quad (7)$$

where we have explicitly separated the inertial, \mathcal{I} , from the dissipative, \mathcal{D} , part of the linear operator and where the non-homogeneous term composed by the forcing and by the second-order passive structure functions is summarized

in the expression, \mathcal{F}_{nq} .

It is useful to highlight in the two-scales correlation function, $P_{n,n+l}$, the dependency from the scale separations by introducing the set of variables, C_l :

$$P_{n,n+l} = C_l P_{n,n}. \quad (8)$$

From basic scaling principle one may argue that the asymptotic scaling behavior is given by the so-called fusion-rules [7, 10, 15, 16]:

$$P_{n,n+l} \sim \frac{\langle |\theta_{n+l}|^2 \rangle}{\langle |\theta_n|^2 \rangle} \langle |\theta_n|^4 \rangle \quad l \rightarrow \infty \quad (9)$$

which means that $C_l \sim C_\infty k_l^{\xi-2}$ for l positive and large enough. Similarly, for l negative, we may write:

$$P_{n,n-l} = D_l P_{n,n} \quad (10)$$

where now the asymptotic behavior of D_l feels the fourth-order scaling behavior: $D_l \sim D_\infty k_l^{-(\xi-2)-\rho_4}$ for l large enough, with $\rho_4 = H(4) - 2H(2)$ being the anomaly of the fourth order scaling exponent. The two sets of variables, D_l and C_l are not independent. By introducing the notation, $x = \lambda^{\xi-2}$ and $R = \lambda^{\rho_4}$ one may rewrite both of them as a function of a new set of variables, Γ_l defined as: $C_l = \Gamma_l x^l$ and $D_l = \Gamma_l / (xR)^l$ [12]. The assumption that fusion rules are satisfied is the only crucial point in computing the zero modes. The existence of fusion rules implies that correlation functions show scaling in the inertial range.

The infinite set of equations for the inertial-range *zero-mode* of (6), $\mathcal{I}_{mq,m'q'} P_{m'q'} = 0$, can be rewritten in the following form:

$$A_0(x, R) + B_{0,1}(x, R)\Gamma_1 = 0 \quad q = n \quad (11)$$

$$A_1(x, R) + B_{1,1}(x, R)\Gamma_1 + B_{1,2}(x, R)\Gamma_2 = 0 \quad n = q + 1 \quad (12)$$

$$B_{n,n-1}(x, R)\Gamma_{n-1} + B_{n,n}(x, R)\Gamma_n + B_{n,n+1}(x, R)\Gamma_{n+1} = 0 \quad n > q + 1, \quad (13)$$

where the functions $A_0, A_1, B_{i,j}$ are known functions of x and $R = \lambda^{\rho_4}$. The computation of the zero modes means to find out the numbers R and Γ_i which solves (11), (12) and (13). Let us remark that for any given total number of shells, N , we have $N + 1$ equations and $N + 2$ unknown which are given by the Γ_i for $i = 1, \dots, N + 1$ plus the parameter directly affected by the fourth-order anomalous exponent, $R(\rho_4)$. Thus, it is impossible to find a solutions unless some extra information is added to our problem. This information can be found by observing that for large n the functions B appearing in (13) become constants independent of both x and R . In the limit of large n , defining

$$Z_n = \frac{\Gamma_{n+1}}{\Gamma_n}$$

one finds that equation (13) can be rewritten as:

$$Z_{n+1} = \Phi(Z_n) \quad (14)$$

where the explicit form of Φ is given in [12]. The map (14) possesses a fixed point $Z^* = 1$ for large shell index, n , which corresponds to the fact that Γ_n reaches a plateau for large n , i.e. to the fact that fusion rules are asymptotically satisfied. The crucial point is to observe that Z^* is a stable fixed point for the inverse of Φ , i.e. for

$$Z_l = \Phi^{-1}(Z_{l+1}). \quad (15)$$

The stability of Z^* for (15) allows us to compute the values of Z_n for small n , i.e. we start with $Z_\infty = 1$ and then we compute Z_m by using (13) up to $m = 2$. In this way we can compute Z_2 as a function of R and x . Thus equations (11) and (12) become:

$$A_0(x, R) + B_{0,1}(x, R)\Gamma_1 = 0 \quad (16)$$

$$A_1(x, R) + B_{1,1}(x, R)\Gamma_1 + B_{1,2}(x, R)Z_2(x, R)\Gamma_1 = 0. \quad (17)$$

Equations (16) and (17) have two unknowns, namely Γ_1 and R , for two equations and, therefore, one can find a solution. The analytical solution turns out to be in perfect agreement with the numerics both for the fourth order object here described [12] and for higher order correlations [13]. This ends the review of the analytical results

previously obtained on the model.

The solution of the Kraichnan shell model for passive scalar provides us the rigorous computation of the zero modes. We want now to understand whether the computation of the zero modes can be pursued by using the concept of time dependent random multiplicative processes. In order to define a suitable TRMP to define the case of the Kraichnan passive scalar we take the usual TRMP discussed in the previous section for the updating of scalar fluctuations at two adjacent scales:

$$\theta_{n+1}(t) = A_{n+1}(t)\theta_n(t), \quad (18)$$

where A_i are i.i.d. random variables with a time-independent probability $P(A)$. The only difference with the TRMP for the velocity field is that now the time upgrading of the multipliers must satisfy the dynamical law: $\partial_t \theta \sim v \partial \theta$. Thus, we need to update the multiplicative structure (18) at the random time interval $\tau_{n+1} = 1/(k_n u_n)$, uncorrelated from the probability distribution of the multipliers them-self (scalars are passive). Moreover, because the advection field is a Gaussian field with correlation functions proportional to $k_n^{-\xi}$ one can deduce that τ_n is not a random time and should be chosen as $\tau_n = k_n^{\xi-2}$. One can show that such a class of TRMP predicts a non trivial behavior of the fusion rules coefficients, C_l, D_l . Moreover, the detailed behavior of the fusion rules coefficients are determined by the spatial intermittency, i.e. the constrain $\partial_t \theta \sim v \partial \theta$ between temporal and spatial scaling induced by the Navier-Stokes structure in a Lagrangian reference frame is satisfied. This is the crucial point we need to use to solve our problem. We can summarize our discussion in the following way. TRMP provide us with a relationship between each fusion rule coefficients, C_l, D_l and the anomalous exponents. In this way we are building a stochastic closure for eqs. (11,12,13). Moreover, the time and spatial dependencies of the stochastic process are consistent with the structure of the deterministic equation of motion. We want here to show that beside the exact method discussed before to find the *zero-mode* also the stochastic closure through the TRMP works.

In the following we assume that the –bare– probability $P(A)$ is log normal. We are aware that log normal probability distributions are not consistent with the anomalous scaling of turbulent flows or shell models for large orders, even for the case of the Kraichnan shell model. However as far as we are interested to compute $H(p)$ for rather small p , log normality is a reasonable approximation which simplifies the analytical computations. Because we know that $H(2) = 2 - \xi$, the probability distribution $P(A)$ depends only on a single unknown parameter σ which describes the variance of the log normal fluctuations. By using the exact solution previously discussed, for each value of ξ we can compute the value of $H(4)$. Thus for each value of ξ we can fix the parameters of the log-normal distribution in order to reproduce the anomalous exponent. We can next simulate the TRMP numerically and compute the value of the fusion rules coefficients, Γ_l . The most sensitive test is made by comparing the prediction on the asymptotic values of Γ_l for large l which we denote by Γ_∞ (notice that $\Gamma_0 = 1$ by definition) as extracted from the computation of the zero-mode and from the TRMP.

Before doing a direct comparison between the TRMP and the exact solution, we need to discuss another subtle point. The definition of a random multiplicative process shows an extra degree of freedom that is not fixed neither by the scaling properties nor by the dynamical scaling. To be more precise, in the case of the Kraichnan model, it is possible to define θ as:

$$\theta_n(t) = g_n \prod_i^n A_i(t)$$

where g_n are i.i.d. random variable for any scale n . Because the probability distribution of g_n does not depend on n , then the scaling properties of θ_n does not depend on g_n . However, the fusion rules coefficients Γ_l do depend on g_n . In particular the quantity Γ_∞ depends on g_n as:

$$\Gamma_\infty(g=1) \rightarrow \Gamma_\infty(g) \frac{\langle g^2 \rangle^2}{\langle g^4 \rangle}$$

Thus it seems that in our way to apply time dependent multiplicative process to the Kraichnan shell model we are not able to fix the fusion rules coefficients. This is rather disappointing because we start all our analysis by pointing out that the shape of the fusion are determined by the time dynamics in a TRMP. However, the function g and its probability distribution should not depend on intermittency itself. In particular, it is relatively easy to compute g and its probability distribution for the Kraichnan model by observing that for $\xi = 2$ all scaling exponents $H(p) = 0$ and $\rho_p = 0$, as already observed in the work by Kraichnan [14]. Using this informations in equations (11,12,13) we find that $\Gamma_1 = 1/2$. Similarly, we can generalize this information for all fusion rules coefficients. This constrains can be satisfied only by a suitable choice of g_n . It turns out that in the Kraichnan shell model this is equivalent to choose g_n to be Gaussian. Thus the value of Γ_∞ should be multiplied by 2 in order to compare it with the TRMP. The comparison is done in figure (1).

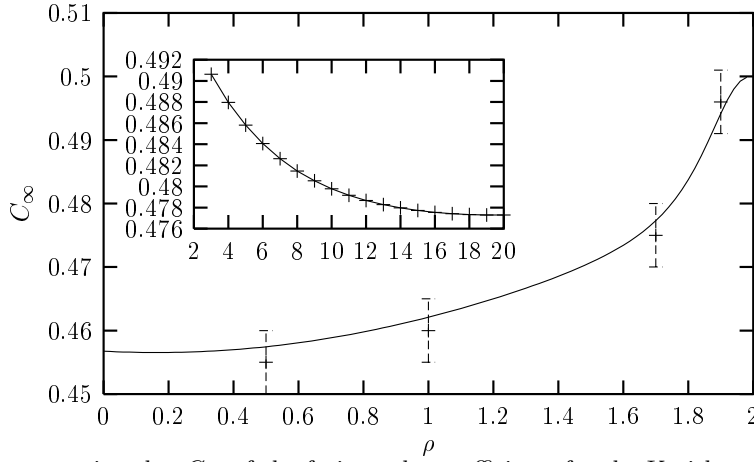


FIG. 1. Results for the asymptotic value C_∞ of the fusion rules coefficients for the Kraichnan model. For different values of ξ , the result obtained by the analytic computation of the zero modes (continuous line) is compared with the estimate (+) obtained using the time dependent random multiplicative process (TRMP). Inset: plot of Γ_n vs. n in the case of the TRMP. The value of Γ_n is multiplied by the factor $\frac{1}{2}$.

As one can see the results are in extremely good agreement with the exact solution. The above results provide us with a complete and clear physical intuition of what a *zero-mode* is. We have shown that the interpretation of anomalous scaling in terms of *zero-modes* is fully compatible with the statistical properties of multiplicative stochastic models. The only missing brick was the importance of temporal dynamics. Anomalous scaling as described by the zero-modes of the inertial operator for the *simultaneous* p -th order structure functions is the outcome of a physical *time-dependent* energy transfer from large scales to small scales. Here we have shown that a suitable closure based on TRMP is indeed sufficient to calculate the zero-mode for fourth-order structure function, $\langle |\theta_m|^2 |\theta_q|^2 \rangle$ in the inertial range.

IV. NON-LINEAR SHELL MODEL

Here we want to understand if a result similar to the one shown in the previous section still holds true for the nonlinear shell models. This is important because: (i) we exploit the possibility to use TRMP in the full non linear case; (ii) we can generalize the concept of zero modes; (iii) we find out a way to compute the scaling exponents. The model we used is an improved version of the GOY model [17, 18], proposed in [19] (see also [4] for a recent review):

$$\left(\frac{d}{dt} + \nu k_n^2 \right) u_n = i (k_n u_{n+1}^* u_{n+2} + b k_{n-1} u_{n+1} u_{n-1}^* + (1+b) k_{n-2} u_{n-2} u_{n-1}) + f_n, \quad (19)$$

where u_n is a complex variable representing the velocity fluctuations at wavenumber k_n , where $k_n = 2^n k_0$. Numerical simulations show that the variables u_n exhibit anomalous scaling for $-1 \leq b \leq 0$, namely:

$$S_p(n) \equiv \langle |u_n^p| \rangle \sim k_n^{-\zeta(p)}, \quad (20)$$

where $\zeta(p)$ is a non linear function of p . Numerically it is observed that the anomalous scaling behavior depends on the parameter b and it does not depend on the specific form chosen for the large scale forcing f_n .

A. Computing the scaling exponents

By defining $Q_n(t) = u_n(t) u_n^*(t)$, we start by searching a solution of the equation obeyed by the simultaneous fourth-order correlation in the limit of zero viscosity, i.e. neglecting dissipative effects:

$$\frac{d}{dt} \langle Q_m Q_n \rangle = k_{n+1} \langle Q_m W_{n+1} \rangle + b k_n \langle Q_m W_n \rangle - (1+b) k_{n-1} \langle Q_m W_{n-1} \rangle + (n \leftrightarrow m) = 0, \quad (21)$$

where we have introduced the flux-variable given by the third order object $W_n = \text{Im}(u_{n+1}^* u_n u_{n-1})$. Equations (21) can be written as an infinite set of linear equations for the fifth order correlation function: $\langle Q_n W_m \rangle$. First, we can pick out the asymptotic behavior given by the usual fusion-rule:

$$\langle Q_{n+l}W_n \rangle = D_l k_l^{-\zeta(2)} S_5(n) \quad \langle W_{n+l}Q_n \rangle = C_l k_l^{-\zeta(3)} S_5(n). \quad (22)$$

Fusion rules are a general properties of the correlation functions in turbulence, predicted by random multiplicative processes and verified with very good accuracy in laboratory experiments [10, 16]. In particular, it is known that for large l , C_l and D_l are no longer dependent on l .

How to obtain informations on the behavior of D_l and C_l ? By restricting ourselves to equal time correlation functions (i.e. equations (1)) there is no hope to close the problem we are facing and it is impossible to get any useful information by using equations (21). In order to make progress, we now assume as in section (III), that the statistical properties of u_n can be described in terms of a time dependent random multiplicative process. We will now employ the following approximations: (i) we use C_l and D_l only for small l , i.e. $l \leq 2$; (ii) for small l we can assume that $C_l = D_l$. Using these approximations we can rewrite from (21) the equations regarding, C_1 and C_2 as follows:

$$\langle Q_n W_n \rangle [C_1 + b - (1+b)C_1 R^{-1}] = 0, \quad (23)$$

$$\langle Q_n W_n \rangle [(1 - (1+b)xR^{-1})C_2 + b(1+x)C_1 + (Rx - (1+b))] = 0, \quad (24)$$

where we have introduced the short hand notation, $x = \lambda^{-\zeta(2)}$, for the dependency on the second order exponents and $R = \lambda^{\zeta(3)+\zeta(2)-\zeta(5)}$ for the dependency on the fifth order anomaly, $\rho_5 = \zeta(3) + \zeta(2) - \zeta(5)$. Also here, as in the passive scalar, we have more unknowns than equations. Precisely, once given the second order exponent, $\zeta(2)$, we have two equations and three unknowns, the two fusion rules coefficients for close-by shells, C_1, C_2 and the fifth-order anomaly, $R(\rho_5)$. Unfortunately one cannot follow the same –winning– strategy adopted for the passive scalar, because here the equivalent of map (14) is not stable for back iteration. In order to close the problem, we must provide informations on the value $Z_2 = C_2/C_1$ as a function of ρ_5 and $\zeta(2)$. Here is where we want to exploit the TRMP.

In order to apply our strategy, we first need to face the following problem. The structure of the equations we want to close (23) deals explicitly with complex shell variables. Therefore one should define two correlated random processes one for the amplitude, $|u_n|$, and another for the phase of the velocity shell variable. Such a strategy, although feasible, introduces new unknowns which need to be fixed either by using the equations of motion or by using additional informations. To not increase the complexity of the problem we look for a simpler and suitable approximation. The key point is that we need to use TRMP just to obtain the quantity $Z_2 = C_2/C_1$, i.e. we need to control the ratio

$$\mathcal{R} = \frac{\langle W_{n+2}Q_n \rangle}{\langle W_{n+1}Q_n \rangle}. \quad (25)$$

Defining $u_n = |u_n| \exp(i\phi)$, we can write:

$$\mathcal{R} = \frac{\langle |u_{n+3}| |u_{n+2}| |u_{n+1}| |u_n|^2 \sin \Delta_{n+2} \rangle}{\langle |u_{n+2}| |u_{n+1}| |u_n|^3 \sin \Delta_{n+1} \rangle}, \quad (26)$$

where $\Delta_n = \phi_n + \phi_{n-1} - \phi_{n+1}$. Expression (26) tells us that, if the correlations between phases and amplitudes is negligible, we can rewrite (26) as follows:

$$\mathcal{R} = \frac{\langle |u_{n+3}| |u_{n+2}| |u_{n+1}| |u_n|^2 \sin \Delta_{n+2} \rangle}{\langle |u_{n+2}| |u_{n+1}| |u_n|^3 \sin \Delta_{n+1} \rangle} \sim \frac{\langle |u_{n+1}|^3 |u_n|^2 \sin \Delta_{n+2} \rangle}{\langle |u_n|^5 \sin \Delta_{n+1} \rangle}, \quad (27)$$

where we have fused the shell variables at scales $n+3$ and $n+2$ with shell variable at scale $n+1$ in the numerator and shell variables $n+2$ and $n+1$ with shell at scale n in the denominator. The above considerations can be formally stated by writing:

$$\frac{\langle W_{n+2}Q_n \rangle}{\langle W_{n+1}Q_n \rangle} = \mathcal{K}(k_n, b) \frac{\langle |u_{n+1}|^3 |u_n|^2 \rangle}{\langle |u_n|^5 \rangle}, \quad (28)$$

where $\mathcal{K}(k_n, b)$ takes into account the correlation, if any, between the phases ($\sin(\Delta_n)$) and the amplitude of the shell variables. We expect that the quantity $\mathcal{K}(k_n, b)$, defined in (28), does not depend on the scale (at least in the inertial range) and might depend on the degree of intermittency, i.e. on b . In particular, if \mathcal{R} strongly depends on the correlation between phases and amplitudes of the shell variables, then \mathcal{K} should be strongly dependent on the free parameter, b , entering in the definition of the non linear terms.

The above discussion can be summarized by saying that the quantity $\mathcal{K}(k_n, b)$ is a direct measure of the importance of the cross-correlations between shell amplitudes and phases with respect to observable based only on shell amplitudes. In order to work out a suitable strategy to apply TRMP as a statistical closure for the non linear shell model, we only need that \mathcal{K} is independent of b . Let us remark that such a requirement is not equivalent to a "random phase

approximation" (which would imply $\mathcal{K} = 1$). In the following we shall assume that \mathcal{K} is independent on intermittency corrections, i.e. on b . Our assumption is justified by the numerical results shown in figure (2). As one can see, the parameter \mathcal{K} is indeed constant, independent of both the shell index and of the intermittency intensity as measured by the variation of the parameter b in the equation of motion.

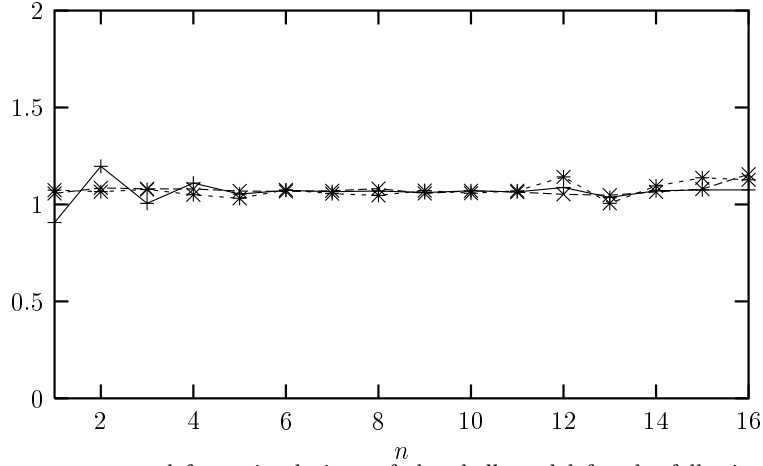


FIG. 2. Plot of $\mathcal{K}(k_n, b)$ vs n computed from simulations of the shell model for the following values of the parameter b : $b = -0.4$ (+), $b = -0.6$ (x), $b = -0.7$ (*). Only data in the inertial range are shown.

Consequently, we may safely proceed with a simple TRMP based on amplitudes only, using equation (28) with $\mathcal{K} = \text{const}$ still to be determined. Concerning the multipliers distributions (18), as in the case of the Kraichnan shell model, we assume that $P(A)$ is log-normal. Let us recall that the exponents, $\zeta^{(s)}(p)$, measured from the scaling of the stochastic signal, $\overline{|u_n|^p} \sim k_n^{\zeta^{(s)}(p)}$ do not coincide with the *bare* scaling exponents as estimated by the instantaneous multiplicative process, $\zeta^{(b)}(p) = -\log_\lambda \langle A^p \rangle$, due to the correlation between the local eddy turn over time and the velocity fluctuations, the time dynamics *re-normalize* the spatial scaling [8]. This is an extra complication with respect to the passive case. Hereafter we always refer to the bare exponents as $\zeta^{(b)}(p)$ and to those actually measured on the stochastic signal as $\zeta^{(s)}(p)$. We proceed by performing the numerical estimate of the scaling properties of the stochastic signal (4) at changing the parameters of the *bare* log-normal distribution, $P(A)$, of the multipliers. Any log-normal distribution is fixed by two -bare- parameters defining the mean and the variance. We fix the mean of the distribution such as the third order exponent measured on the TRMP is consistent with the 4/5 law, $\zeta^{(s)}(3) = 1$. Now, we are left with only the variance, σ , of the multipliers probability distribution, $P(A)$, as a free parameter. In order to have a control on the ratio:

$$C_2/C_1 = \mathcal{K} \frac{\langle |u_{n+1}|^3 |u_n|^2 \rangle}{\frac{\langle |u_{n+1}|^3 \rangle}{\langle |u_n|^3 \rangle} \langle |u_n|^5 \rangle}$$

we may estimate the unknowns multi-scale correlation functions appearing in the r.h.s. by using the TRMP at varying the log-normal distribution:

$$\frac{\langle |u_{n+1}|^3 |u_n|^2 \rangle}{\frac{\langle |u_{n+1}|^3 \rangle}{\langle |u_n|^3 \rangle} \langle |u_n|^5 \rangle} \sim \frac{\overline{|u_{n+1}|^3 |u_n|^2}^{(\sigma)}}{\frac{\overline{|u_{n+1}|^3}^{(\sigma)}}{\overline{|u_n|^3}^{(\sigma)}} \overline{|u_n|^5}^{(\sigma)}}, \quad (29)$$

where we have added a superscript (σ) in the averages from the TRMP to remind the dependency on the variance of the log-normal distribution.

As a result we have a guess on the ratio C_2/C_1 at varying σ up to the still unknown constant \mathcal{K} .

The results of the numerical simulations are shown in figure (3) where we plot (29) as a function of

$$\rho_5(\sigma) = \zeta^{(s)}(3) + \zeta^{(s)}(2) - \zeta^{(s)}(5)$$

in the TRMP.

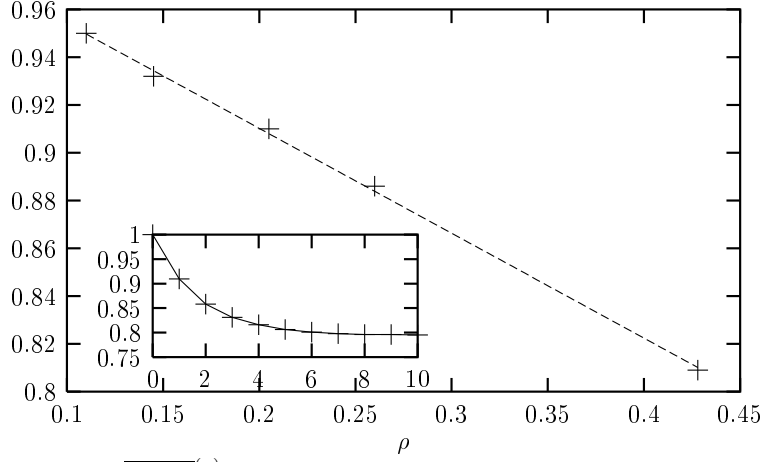


FIG. 3. Results for $\frac{\overline{|u_{n+1}|^3|u_n|^2}^{(\sigma)}}{\frac{\overline{|u_{n+1}|^3}^{(\sigma)}}{\overline{|u_n|^3}^{(\sigma)}}\overline{|u_n|^5}^{(\sigma)}}$ as a function of $\rho_5(\sigma) = \zeta^{(s)}(3) + \zeta^{(s)}(2) - \zeta^{(s)}(5)$. We plot the data measured using the TRMP (+) and the best linear fit $1 - 0.44\rho_5$. Inset: typical behavior of the fusion rule coefficients, C_n , vs n obtained from a TRMP stochastic signal.

As one can easily see from figure (3), expression (29) is extremely well fitted by a linear behavior:

$$\frac{\overline{|u_{n+1}|^3|u_n|^2}^{(\sigma)}}{\frac{\overline{|u_{n+1}|^3}^{(\sigma)}}{\overline{|u_n|^3}^{(\sigma)}}\overline{|u_n|^5}^{(\sigma)}} = 1 - 0.44\rho_5(\sigma). \quad (30)$$

This is the third equation, linking the unknowns in (23) and (24) and closing the problem. It has been obtained by using the TRMP. This is not yet the end of the story. It is not enough to plug the numerical result (30) into eqs. (23) and (24) to close consistently the equations. The problem is connected to the possible presence of a *re-normalizing* scale-independent stochastic variable in the multiplicative process – the g_n variable already discussed for the passive scalar case. We already discussed that the presence of such a scale-invariant distribution changes only one overall constant in the multi-scale behavior. Moreover, we already know that another unknown constant overall, \mathcal{K} , shows up due to the phases' statistics. Summing the two effects, we can assume that the *true* ratio C_2/C_1 which must be plugged in the equation of motion can be estimated by the result of the TRMP (30) plus a multiplicative, unknown, constant, \mathcal{D} independent on the intermittency of the model:

$$\frac{C_2}{C_1} = (1 - 0.44\rho_5(\sigma)) \mathcal{D}. \quad (31)$$

Using equations (23,24,31), we can compute the fifth-order anomaly, $\rho_5 = \log_\lambda(R)$, by solving the system of three equations in three unknown, C_2, C_1, ρ_5 :

$$C_1 + b - (1 + b)C_1R^{-1}(\rho_5) = 0 \quad (32)$$

$$(1 - (1 + b)xR^{-1}(\rho_5))C_2 + b(1 + x)C_1 + (R(\rho_5)x - (1 + b)) = 0 \quad (33)$$

$$C_2/C_1 = \mathcal{D}(1 - 0.44\rho_5) \quad (34)$$

To our knowledge, there are no simple theoretical arguments which can be used in order to fix the value of \mathcal{D} . We fix it by assuming that for $b = -0.4$ we recover the value of ρ_5 computed in the numerical simulations. It turns out that $\mathcal{D} = 0.85$. We can then compute ρ_5 for all values of b in the range $-1 < b < 0$. In Figure (4) we show the computation of ρ_5 obtained by using (32-34) together with the numerical estimate of ρ_5 obtained by simulations of the shell model. As one can see the results are in very good agreement with the numerical data *for the whole range of b* .

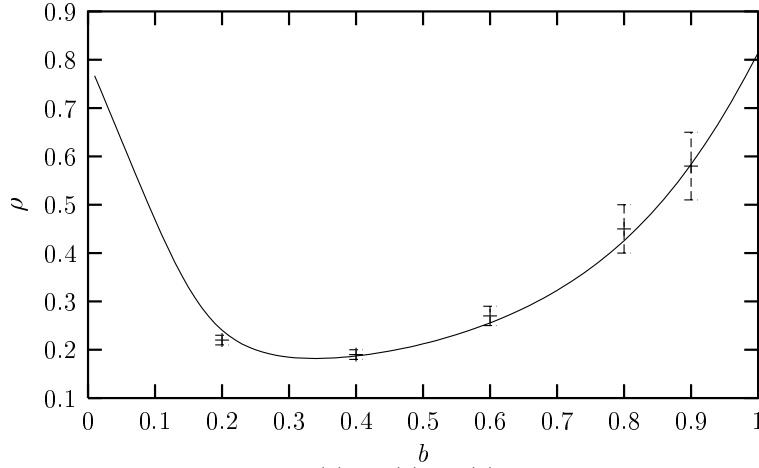


FIG. 4. The values of the fifth order anomaly $\rho_5 = \zeta(3) + \zeta(2) - \zeta(5)$ as a function of b obtained from the TRMP closure approach (continuous line) are compared with the numerical estimate coming from simulations of the shell model (+).

In order to validate our results, we have compared the estimate of the anomalous anomaly ρ_5 for the values of b and λ corresponding to the curve

$$\lambda = \frac{1}{1+b}. \quad (35)$$

The curve (35) is determined by the requirement that the second in-viscid invariant, beside the total energy, $\sum_n u_n^2$, keeps the physical dimensions of Helicity, $\sum_n (-)^n k_n |u_n|^2$ [20, 21]. It is known [22] that along this curve intermittency stay constant. Numerical estimate of (30) for different values of λ do not show any appreciable difference with respect to what plotted in figure (3). Thus, we can still use (34) as a numerical estimate of C_2/C_1 with *the same value of \mathcal{D}* . In fig. (5) we show the comparison between the results of the closure on the special curve (35) and the values estimated by numerical simulations. Again, the stochastic closure works perfectly, allowing for a precise determination of the anomaly along this curve.

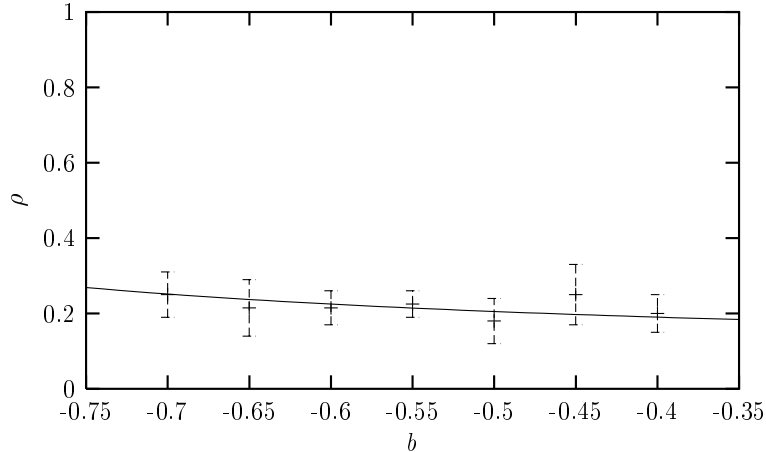


FIG. 5. The values for the fifth-order anomaly $\rho_5 = \zeta(3) + \zeta(2) - \zeta(5)$ as a function of b obtained from the TRMP closure approach along the curve $\lambda = \frac{1}{1+b}$ (continuous line) are compared with the numerical estimate coming from simulations of the shell model (+).

V. CONCLUSIONS AND DISCUSSIONS

In this paper we have discussed a possible strategy to compute the scaling exponent of a non linear shell model. The main idea of this strategy is to assume that the statistical properties of the shell variables u_n can be described in terms of a time dependent random multiplicative process. A mathematical way to summarize this idea is the following. Let us define the random variable A_n as:

$$|u_{n+1}| = A_n |u_n|. \quad (36)$$

Let us also assume that $A_n = \lambda^{h_n}$, where the random variables h_n fluctuates with probability distribution $P_n = Z_n \cdot (\exp -V(h_n))$. If we neglect time dynamics the probability distribution of the shell variables is given by the product $\Pi_k P_k$. Time dependent random multiplicative process provides well defined correlations among the random variables for different scales. One can therefore write:

$$P(u_1, u_2, \dots, u_n, \dots) = Z \cdot \exp[-\Sigma_n V(h_n) + \Sigma_{i,j} R_{ij} h_i h_j]. \quad (37)$$

Once $V(h_n)$ is defined, the time dependency selects an unique value of $R_{i,j}$. Formally we can write $R_{i,j} = \Gamma_{i,j}[V]$. It follows that in order to compute the scaling exponents we need to compute $V(h_n)$. The role of $R_{i,j}$ is crucial because it determines the full shape of the coefficients needed to compute the fusion rules. Using the time average equation of motions for the structure functions, we are therefore able to obtain a functional equation for V , whose solutions provides the anomalous scaling exponents of the shell model. In order to understand whether our strategy is providing reasonable results, we have assumed that $V(x)$ is a quadratic function of x . Thus, because $\zeta(3) = 1$, we have only one unknown to be computed corresponding to the quadratic non linearity. With this assumption, the functional equation for V reduces to an equation for one unknown (the quadratic non linearity) which we have solved. The results we obtain are in very good agreement with the numerical simulation of the shell model.

We want to stress that the same procedure can be applied for the passive scalar advected by a random velocity field (the Kraichnan model) with extremely good results and without any ad hoc approximation.

We want to highlight few points which we believe are true independent of the approximations we did in this paper.

(i) The computations of the scaling exponents are feasible if the fusion rules coefficient are known as a function of intermittency (i.e. the function $D(h)$ in the multi-fractal language). (ii) The fusion rules coefficient depends on intermittency because of time dynamics. (iii) Time dependent random multiplicative processes are consistent with the dynamical deterministic structure of the equation of motion and provide an useful tool to compute fusion rules coefficients.

Nevertheless, it is not clear yet, if the stochastic process successfully applied here to close the equation of motion of fourth-order velocity correlation is also the optimal solution for higher order correlation functions. In other words, one has to face also the possibility that different fluctuations (controlling higher order correlation functions) are described by different stochastic processes.

In applying our strategy to the shell model we have performed a number of approximations. In particular we consider an important point to generalize our approach in order to properly take into account the phase dynamics in the shell model. Also, the approximation $C_l = D_l$ should be considered as a first order of approximation in order to develop a systematic theory. We also want to highlight two important topics for future research. First of all, we think that it is important to apply our method in the case of the Kraichnan model in two or more space dimensions. In order to perform such a task we need to develop the field theory of time dependent random multiplicative processes. Also, we need to have a more detailed analytical control of time dependent random multiplicative processes, following the ideas already discussed in [8, 9].

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